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Set Theory

Q.1. Prove that every well ordered set is a linearly ordered set and every linearly ordered set is a partially ordered set. Show ~~these~~ also that the converses do not hold.

Proof: Let $x, y \in X$. Then, well-ordered set $\{x, y\}$, there must be a first element in $\{x, y\}$ which is unique.

Let us suppose that $x, y, z \in X$. Then, $\{x, y, z\} \subseteq X$ so that $\{x, y, z\}$ is well-ordered.

Let us suppose that $x, y \in X$ and $x \neq y$, then $\{x, y\} \subseteq X$. Also, under $\{x, y\}$ will have a unique first element. If the first element is x then, if y is the first element ~~then~~ Hence, for every $x, y \in X$, $(x \neq y)$. Then, $\{x, y\}$ is a linearly ordered set.

In any partially ordered set, each linearly ordered subset is called a chain.

The set $(X, \leq)$ is then ~~called~~ a partially ordered set. For simlicity $(X, \leq)$ is written as X.

For, $X = \{1, 2, 3\}$, then $\subseteq$ in X
$= \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is a chain.

Q. 2. State and prove Schroeder-Bernstein Theorem.

Soln: Statement:

If $X \simeq Y_1 \subset Y$ and $Y \simeq X_1 \subset X$

then, $X \approx Y$.

Proof: Let us suppose that $f: X \simeq Y_1$ and $g: Y \simeq X_1$ are two function which are one-one and onto.

We express this as

$$X \overset{f}{\simeq} Y_1 \text{ and } Y \overset{g}{\simeq} X_1$$

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Rest part of Q.2.

Let $U_0 \approx X - X_1 = X \cap X_1^c$.

then $U_0 \subset X$ and $U_0 \subset X_1^c$ ——— (I)

Suppose the one-one onto map

$$X \overset{f}{\approx} Y_1$$

It gives $U_0 \overset{f}{\approx} V_1 \subset Y_1 \subset Y$ ——— (II)

And the one-one onto map

$$Y \overset{g}{\approx} X_1$$

It gives $V_1 \overset{g}{\approx} U_1 \subset X_1 \subset X$ ——— III

If we continue this process then, we get

$$U_0 \overset{f}{\approx} V_1 \overset{g}{\approx} U_1 \overset{f}{\approx} V_2 \overset{g}{\approx} U_2 \overset{f}{\approx} \text{———} IV$$

In this continuation IV, the sets

$$U_0, U_1, U_2, \cdots U_K, \text{———} (V)$$

are all subsets of X and the sets

$$V_1, V_2, V_3, \cdots V_K, \cdots \text{———} (VI)$$

are all subsets of Y.

In both the sequences V and VI, the sets are disjoint two by two.

For (I) gives, $U_0 \subset X_1^c$ and III gives $U_1 \subset X_1$ giving

$$U_0 \cap U_1 = \phi \text{———} VII$$

Since, IV gives $U_0 \overset{f}{\approx} V_1$ and $U_1 \overset{f}{\approx} V_2$

(VII) gives $V_1 \cap V_2 = \phi$ ——— VIII

Again from IV, $V_1 \overset{g}{\approx} U_1$ and $V_2 \overset{g}{\approx} U_2$

and VIII gives $U_1 \cap U_2 = \phi$

Since $U_0 \subset X_1^c$ and $U_4 \subset X_1$, we have $U_0 \cap U_4 = \phi$

we write $U = U_0 + U_1 + U_2 + \cdots + U_k + \cdots$

and $V = V_1 + V_2 + \cdots + V_k + \cdots$

Then, $U \subset X$ and $V \subset Y$

Proved

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Differential Calculus, B.Sc. part-II (H), paper-III
(partial differentiation)

Q.1. Verify Euler's Theorem when $u = \frac{x(x^3-y^3)}{x^3+y^3}$

Soln: We have, $\log u = \log x + \log(x^3-y^3) - \log(x^3+y^3)$

Differentiating partially with respect to x, we get

$\therefore \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{3x^2}{x^3-y^3} - \frac{3x^2}{x^3+y^3}$

$\therefore \frac{1}{u} x\frac{\partial u}{\partial x} = 1 + \frac{3x^3}{x^3-y^3} - \frac{3x^3}{x^3+y^3}$ ——— (I)

Also, $\frac{1}{u}\frac{\partial u}{\partial y} = \frac{-3y^2}{x^3-y^3} - \frac{3y^2}{x^3+y^3}$

$\therefore \frac{1}{u} y\frac{\partial u}{\partial y} = \frac{-3y^3}{x^3-y^3} - \frac{3y^3}{x^3+y^3}$ ——— II

Adding I and II, we get

$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 1 + \frac{3\cancel{(x^3-y^3)}}{\cancel{x^3-y^3}} - \frac{3\cancel{(x^3+y^3)}}{\cancel{x^3+y^3}} = 1+\cancel{3}-\cancel{3} = 1$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

As the given function $u = \frac{x^4\left(1-\frac{y^3}{x^3}\right)}{x^3\left(1+\frac{y^3}{x^3}\right)} = x\frac{\left(1-\frac{y^3}{x^3}\right)}{\left(1+\frac{y^3}{x^3}\right)} = x\phi\left(\frac{y}{x}\right)$

is a homogeneous function of degree one, therefore u satisfies Euler's Theorem.

Q.2. Verify Euler's Theorem when $u = x^3 \log\frac{y}{x}$

Soln We have $u = x^3(\log y - \log x)$

Differentiating partially w.r.to x, we get

$\frac{\partial u}{\partial x} = 3x^2(\log y - \log x) - x^3.\frac{1}{x}$

$\therefore x\frac{\partial u}{\partial x} = 3x^3(\log y - \log x) - x^3$ ——— (I)

Also, $\frac{\partial u}{\partial y} = x^3.\frac{1}{y} \Rightarrow y\frac{\partial u}{\partial y} = x^3$ ——— II

Adding I and II, we get

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3(\log y - \log x) = 3u$

Now, $u = x^3 \log\frac{y}{x} = x^3\phi\left(\frac{y}{x}\right)$

Here, u is a homogeneous function of degree 3, which is satisfies Euler's Theorem

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Q.3. Verify Euler's Theorem for the expression $x^n \sin \frac{y}{x}$.

Soln Let $u = x^n \sin \frac{y}{x}$

Differentiating partially w.r.to x, we get

$$\frac{\partial u}{\partial x} = n x^{n-1} \sin \frac{y}{x} - x^n \cos \frac{y}{x} \cdot \frac{y}{x^2}$$

$$\therefore x\frac{\partial u}{\partial x} = n x^n \sin \frac{y}{x} - x^{n-1} y \cos \frac{y}{x} \quad \text{——— (I)}$$

Also, $\frac{\partial u}{\partial y} = x^n \cos \frac{y}{x} \cdot \frac{1}{x}$

$$\therefore y\frac{\partial u}{\partial y} = y x^{n-1} \cos \frac{y}{x} \quad \text{——— (II)}$$

Adding I and II, we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n x^n \sin \frac{y}{x} = nu$

The given function $x^n \sin \frac{y}{x}$ is a homogeneous function of degree n.

Hence, the given function satisfies Euler's Theorem.

Q.4. If A, B, C be the angles of a triangle such that

$$\sin^2 A + \sin^2 B + \sin^2 C = \text{constant},$$

prove that $\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}$

Soln We know that in a triangle, $A+B+C = \pi$

Differentiating with respect to B, we get

$$\frac{dA}{dB} + 1 + \frac{dC}{dB} = 0 \Rightarrow \frac{dC}{dB} = -\left(1 + \frac{dA}{dB}\right) \quad \text{——— (I)}$$

By question, $\sin^2 A + \sin^2 B + \sin^2 C = \text{constant}$.

Differentiating with respect to B, we get

$$\sin 2A \cdot \frac{dA}{dB} + \sin 2B + \sin 2C \cdot \frac{dC}{dB} = 0$$

By (I), we get $\sin 2A \cdot \frac{dA}{dB} + \sin 2B - \sin 2C \left(1 + \frac{dA}{dB}\right) = 0$

$$\Rightarrow \frac{dA}{dB}\left(\sin 2A - \sin 2C\right) = \sin 2C - \sin 2B$$

$$\Rightarrow \frac{dA}{dB}\left[2\cos\frac{2(A+C)}{2} \sin\frac{2(A-C)}{2}\right]$$

$$= 2\cos\frac{2(C+B)}{2} \sin\frac{2(C-B)}{2}$$

$$\Rightarrow \frac{dA}{dB} \cdot \cos(A+C) \sin(A-C) = \cos(C+B) \sin(C-B)$$

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Differential Calculus

rest part of Q.4. $\Rightarrow \frac{dA}{dB} \cdot \cos(\pi-B)(\sin A \cdot \cos C - \cos A \cdot \sin C)$

$$= \cos(\pi-A)(\sin C \cos B - \cos C \sin B)$$

$\Rightarrow -\frac{dA}{dB} \cos B (\sin A \cos C - \cos A \cdot \sin C)$

$$= -\cos A (\sin C \cos B - \cos C \sin B)$$

Dividing both sides by $\cos A \cos B \cos C$, we get

$$\frac{dA}{dB} (\tan A - \tan C) = \tan C - \tan B$$

Hence, $\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}$

Q.5. If $u = F(x-y, y-z, z-x)$

and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ all exist, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln: We have, $u = F(x-y, y-z, z-x)$ —— (I)

Let $X = x-y$, $Y = y-z$ and $Z = z-x$

$\therefore \frac{\partial X}{\partial x} = 1, \frac{\partial X}{\partial y} = -1, \frac{\partial X}{\partial z} = 0$

$\frac{\partial Y}{\partial x} = 0, \frac{\partial Y}{\partial y} = 1, \frac{\partial Y}{\partial z} = -1, \frac{\partial Z}{\partial x} = -1, \frac{\partial Z}{\partial y} = 0, \frac{\partial Z}{\partial z} = 1.$

$\therefore$ (I) transforms into $u = F(X, Y, Z)$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial F}{\partial Z} \cdot \frac{\partial Z}{\partial x}$

$$= \frac{\partial F}{\partial X} \cdot 1 + \frac{\partial F}{\partial Y} \cdot 0 + \frac{\partial F}{\partial Z}(-1) = \frac{\partial F}{\partial X} - \frac{\partial F}{\partial Z} \text{ —— (II)}$$

Again, $\frac{\partial u}{\partial y} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial F}{\partial Z} \cdot \frac{\partial Z}{\partial y}$

$$= \frac{\partial F}{\partial X}(-1) + \frac{\partial F}{\partial Y} \cdot 1 + \frac{\partial F}{\partial Z} \cdot 0 = \frac{\partial F}{\partial Y} - \frac{\partial F}{\partial X} \text{ —— III}$$

Also, $\frac{\partial u}{\partial z} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial F}{\partial Z} \cdot \frac{\partial Z}{\partial z}$

$$= \frac{\partial F}{\partial X} \cdot 0 + \frac{\partial F}{\partial Y}(-1) + \frac{\partial F}{\partial Z} \cdot 1 = \frac{\partial F}{\partial Z} - \frac{\partial F}{\partial Y} \text{ —— IV}$$

Adding II, III and IV, we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ proved.

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Q.6 If z and u be functions of x and y defined by $\{z-\phi(u)\}^2 = x^2(y^2-u^2)$, $\{z-\phi(u)\}\phi'(u) = ux^2$ prove that $\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = xy$

Soln: Since, z is a function of x and y.

$$\therefore\ \partial z = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{— I}$$

It is given that $\{z-\phi(u)\}^2 = x^2(y^2-u^2)$ — II

and $\{z-\phi(u)\}\phi'(u) = ux^2$ — III

Taking differentials of both sides of II, we get

$$2\{z-\phi(u)\}\{dz-\phi'(u)\,du\} = 2x\,dx(y^2-u^2)+x^2(2y\,dy-2u\,du)$$

$$\Rightarrow \{z-\phi(u)\}dz - \{z-\phi(u)\}\phi'(u)du = x\,dx(y^2-u^2)+x^2(y\,dy-u\,du)$$

$$\Rightarrow \{z-\phi(u)\}\left\{\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right\} - ux^2du$$

$$= x\,dx(y^2-u^2)+x^2y\,dy-ux^2du$$

[from I and III]

$$\Rightarrow \{z-\phi(u)\}\frac{\partial z}{\partial x}dx + \{z-\phi(u)\}\frac{\partial z}{\partial y}dy = x(y^2-u^2)dx + x^2y\,dy$$

Equating the coefficients of dx and dy separately, we get

$$\{z-\phi(u)\}\frac{\partial z}{\partial x} = x(y^2-u^2) \quad \text{— IV}$$

$$\text{and } \{z-\phi(u)\}\frac{\partial z}{\partial y} = x^2y \quad \text{— V}$$

Multiplying IV and V we get

$$\{z-\phi(u)\}^2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = x^3y(y^2-u^2)$$

$$\Rightarrow \cancel{x^2(y^2-u^2)}\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial y} = \cancel{x^3}y\cancel{(y^2-u^2)}$$

Hence, $\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = xy$ proved